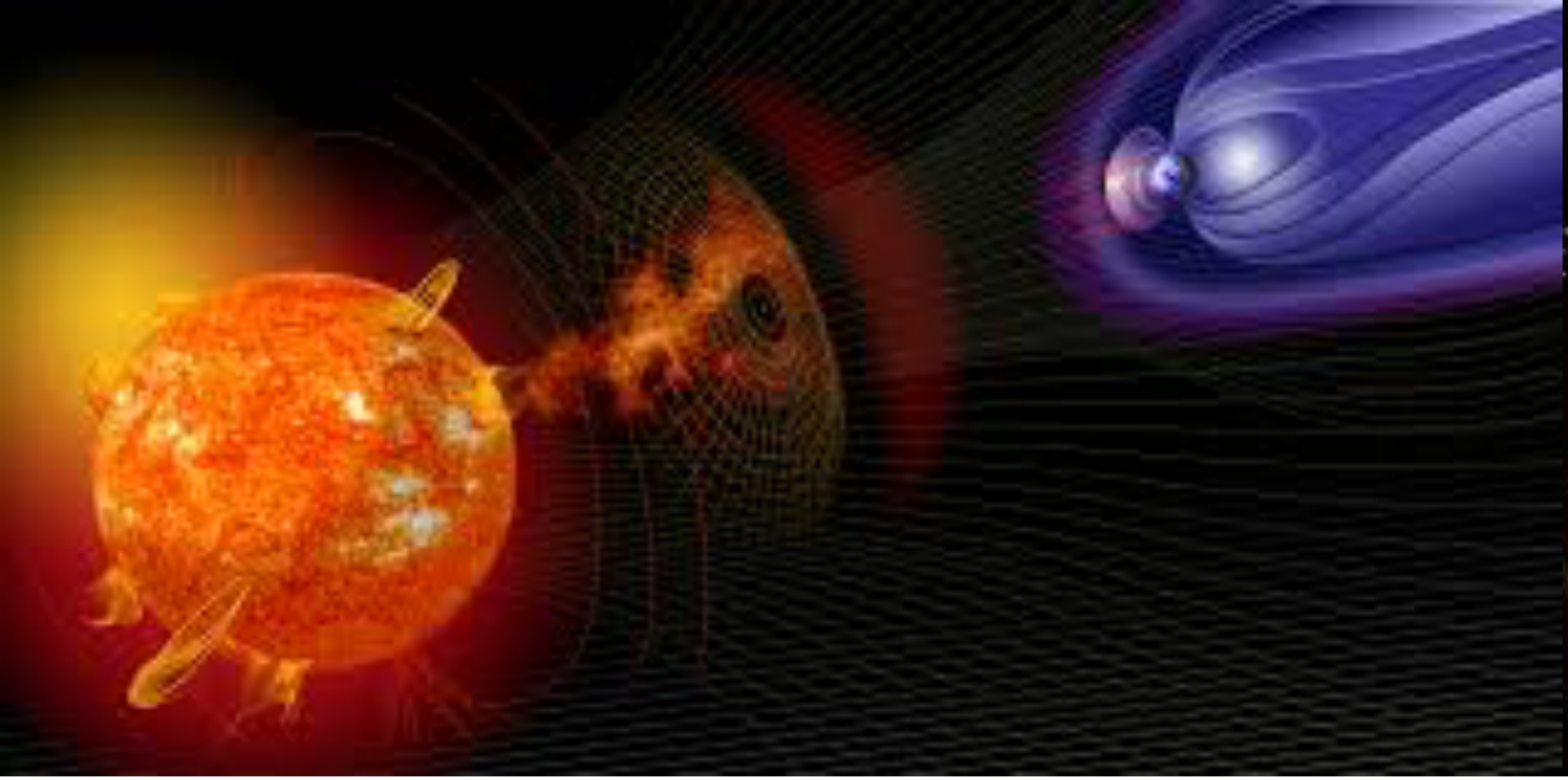
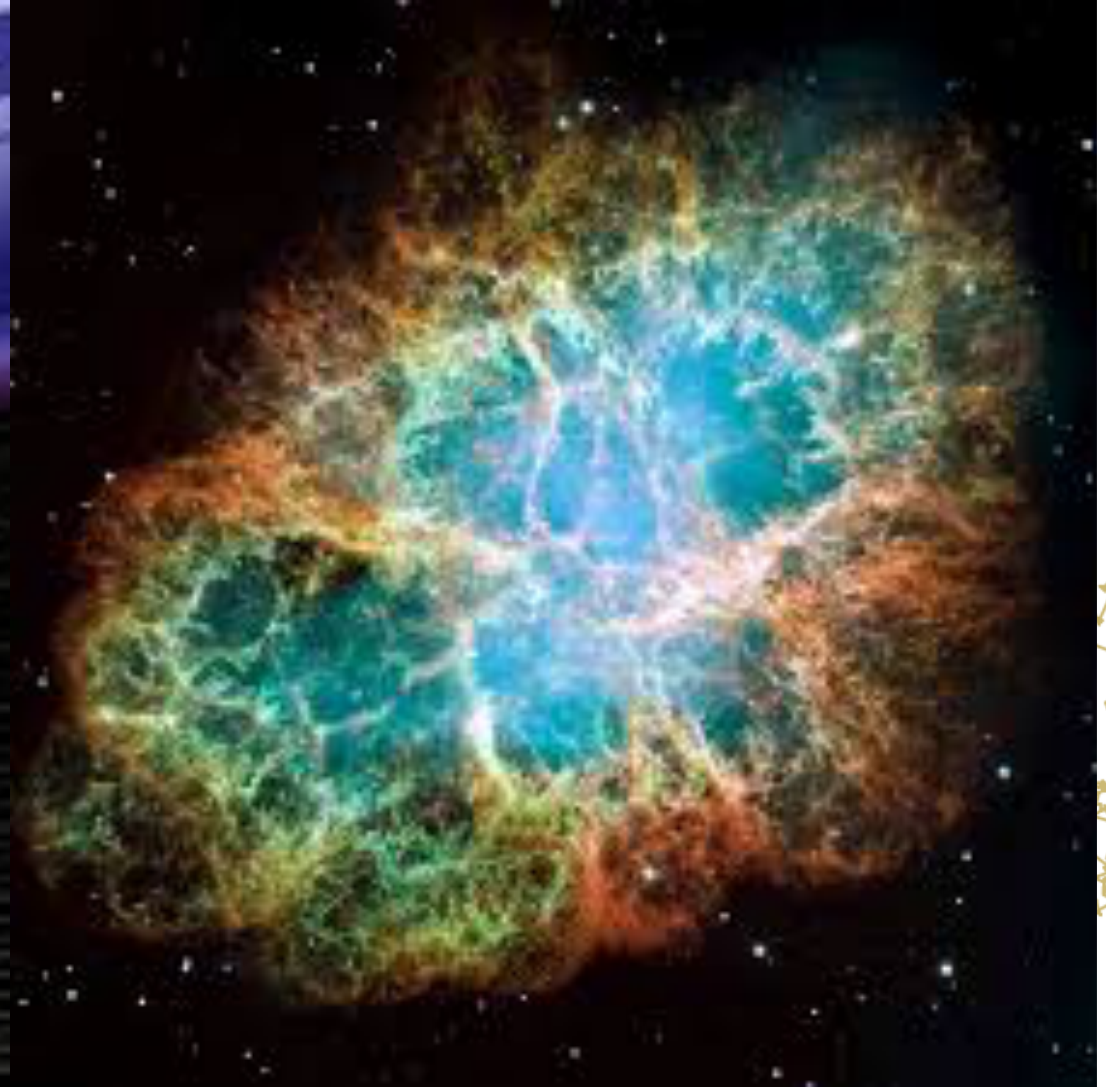
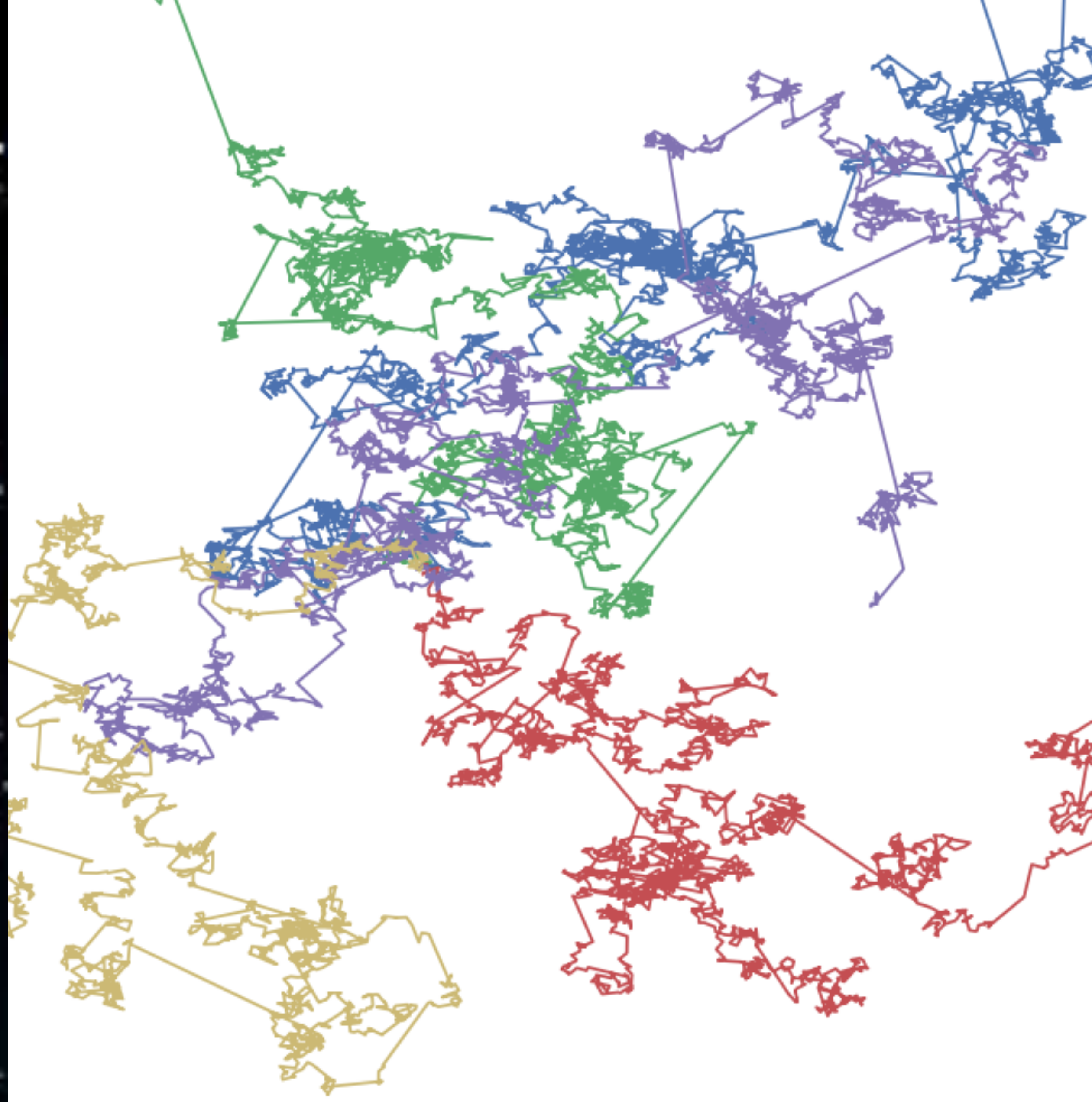




Stochastic Methods for Anomalous Transport and Acceleration of Energetic Particles

ICRC 2021 Berlin,

Frederic Effenberger, Horst Fichtner, Dominik Walter

Ruhr-University Bochum, Germany

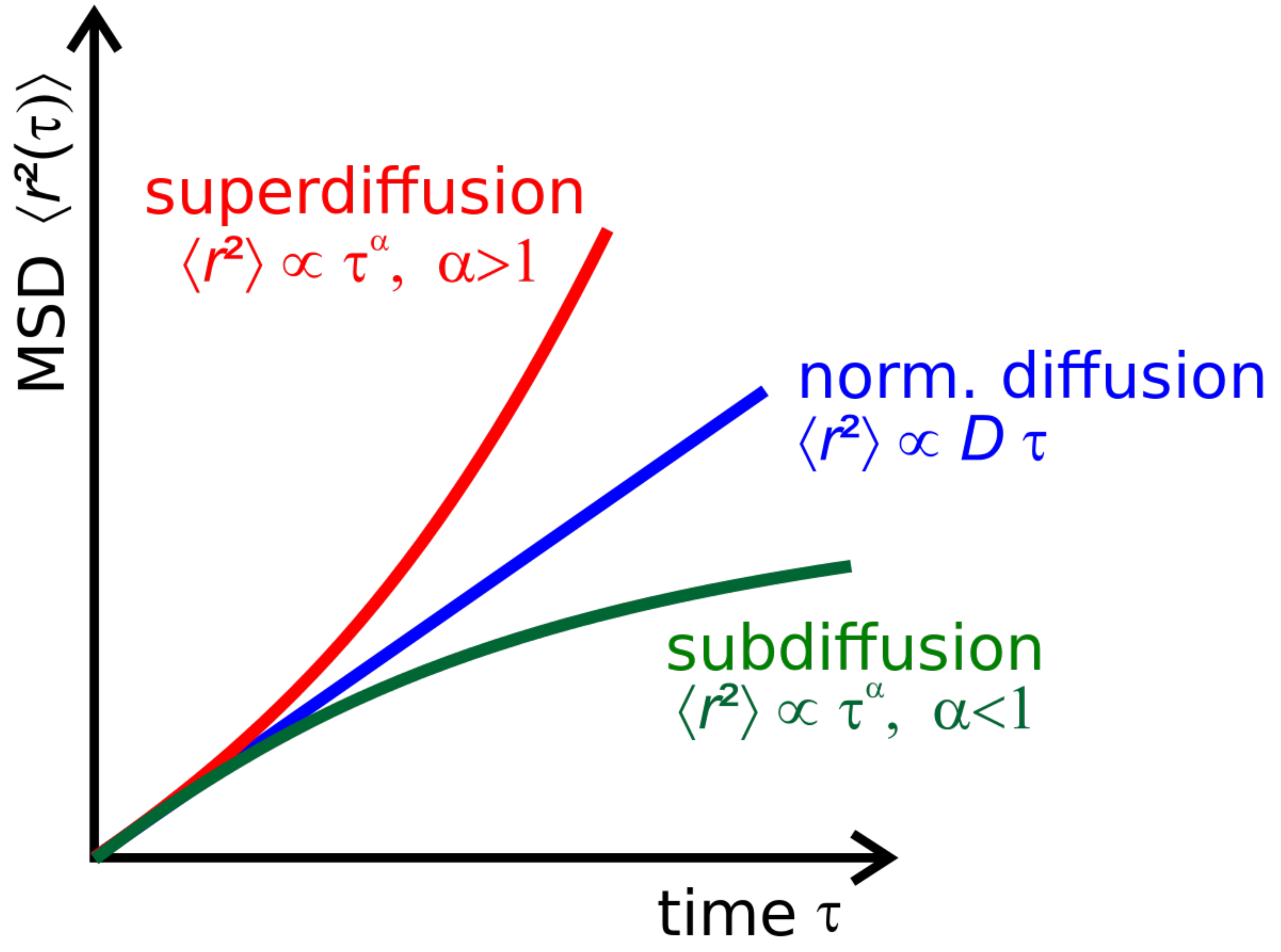
Stochastic Methods for Anomalous Transport and Acceleration of Energetic Particles

- Introduction
- Models for Anomalous Diffusion
  - Fractional Derivatives
  - Non-Gaussian Random Processes
- Applications
  - Shocks
  - Acceleration
  - Transport
- Summary

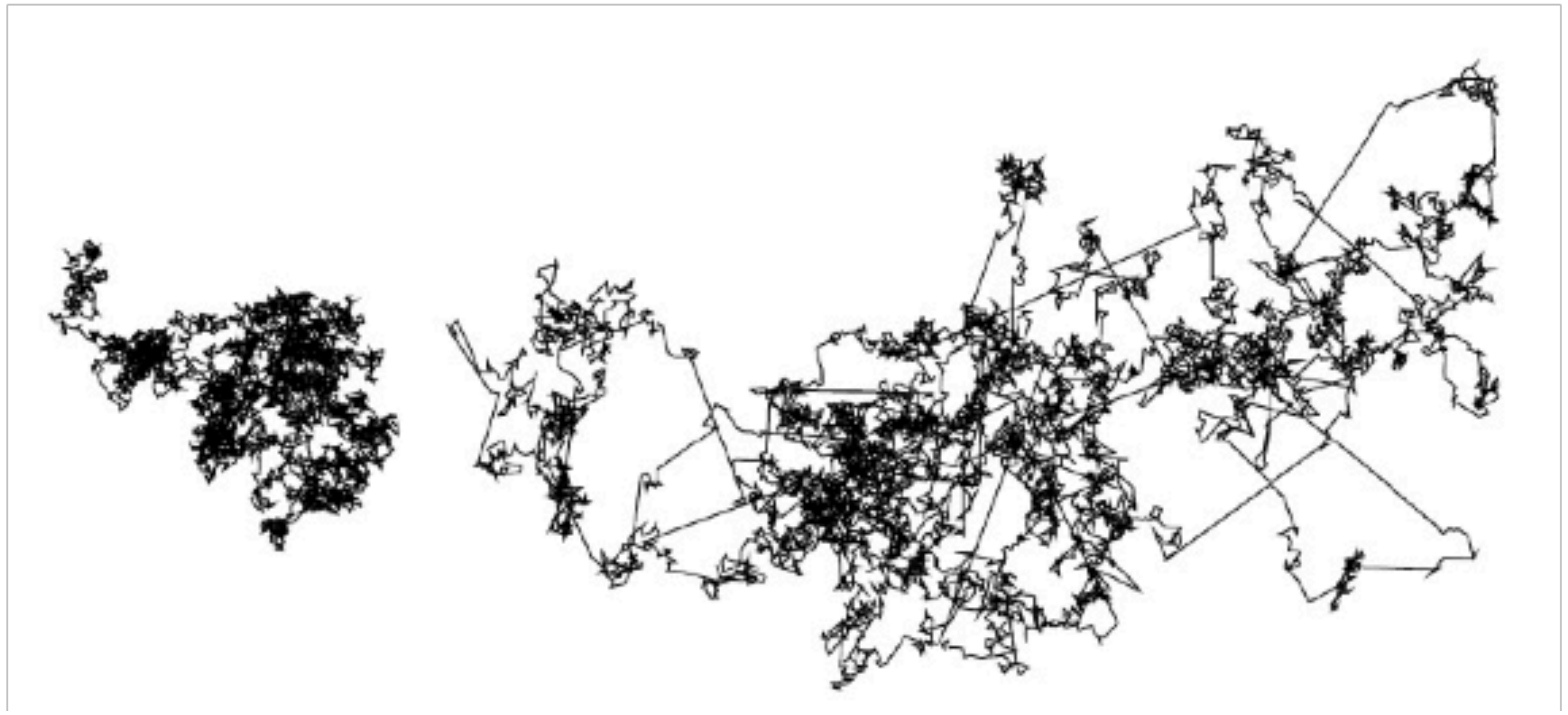
New project at Ruhr-University Bochum,
funded by Deutsche Forschungsgemeinschaft

Anomalous Diffusion

Gaussian	Anomalous	
$(\Delta x)^2 \propto t$	$(\Delta x)^2 \propto t^\zeta$	Superdiffusion: $1 < \zeta < 2$ Subdiffusion: $0 < \zeta < 1$



Subdiffusion (extended waiting times)



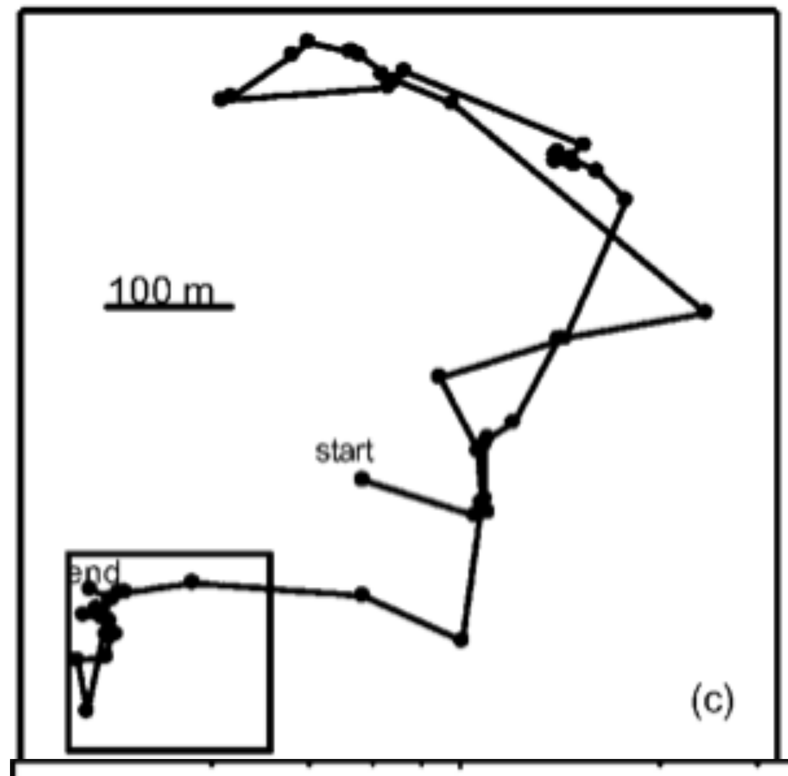
Superdiffusion (Lévy-Flights)

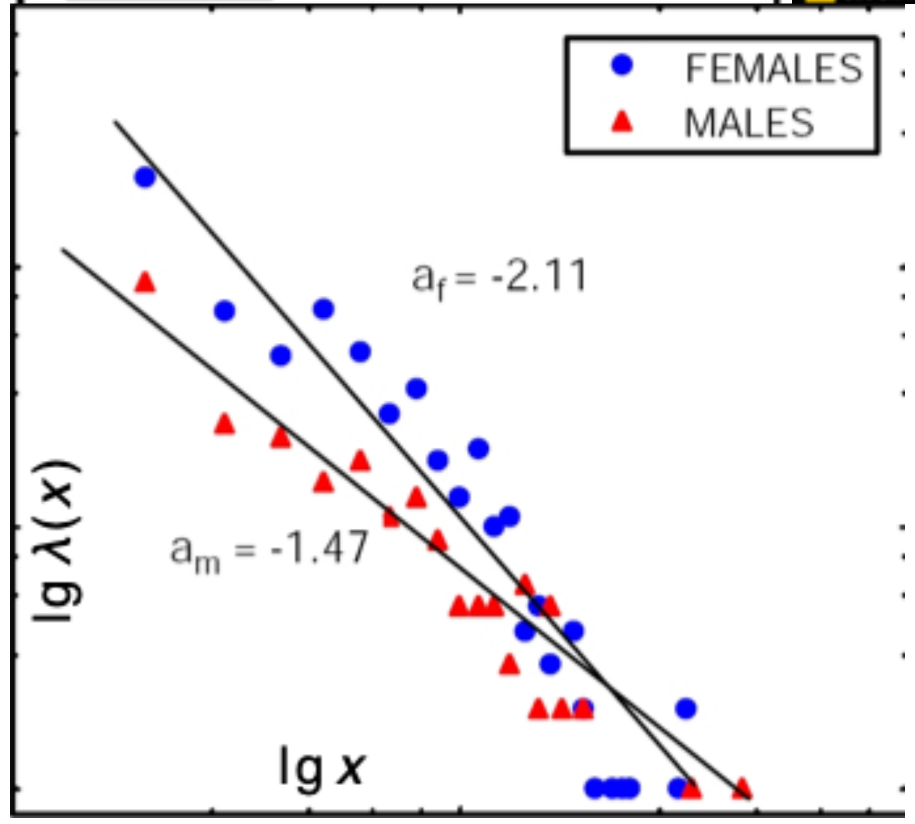
Anomalous Diffusion

Human Travel Behaviour

[Brockmann, 2010]

Jumps of Spider Monkeys





[Ramos-Fernandez et al., 2003]

Models for Anomalous Diffusion

$$(\Delta x)^2 \propto t^\zeta$$

Idea: Generalize Diffusion Equation to non-integer derivatives

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^\alpha f}{\partial |x|^\alpha} + a \frac{\partial f}{\partial x} + \delta(x)$$

Using symmetric fractional Riesz derivative (generalized Laplacian)

$$\frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} = \frac{1}{\pi} \sin\left(\frac{\pi}{2}\alpha\right) \Gamma(1+\alpha) \times \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\alpha}} d\xi$$

$$f(x,t) = \sum_{n=1}^{\infty} \Bigg\{ (1+(-1)^{n+1}) \Bigg[\left(\frac{n\pi}{L}\right)^\alpha \kappa L \cos\left(\frac{n\pi x}{L}\right) - n\pi a \sin\left(\frac{n\pi}{L}x\right) - \left(\frac{n\pi}{L}\right)^\alpha \kappa L \cos\left(\frac{n\pi}{L}(x+at)\right) \exp\left(-\left(\frac{n\pi}{L}\right)^\alpha \kappa t\right) + n\pi a \sin\left(\frac{n\pi}{L}(x+at)\right) \exp\left(-\left(\frac{n\pi}{L}\right)^\alpha \kappa t\right) \Bigg] \Bigg/ \left(\left(\frac{n\pi}{L}\right)^{2\alpha} \kappa^2 L^2 + n^2 \pi^2 a^2\right) \Bigg\}. \qquad (31)$$

[Stern et al., 2014]

Models for Anomalous Diffusion

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ANALYTICAL SOLUTIONS OF A FRACTIONAL DIFFUSION-ADVECTION EQUATION FOR SOLAR COSMIC-RAY TRANSPORT

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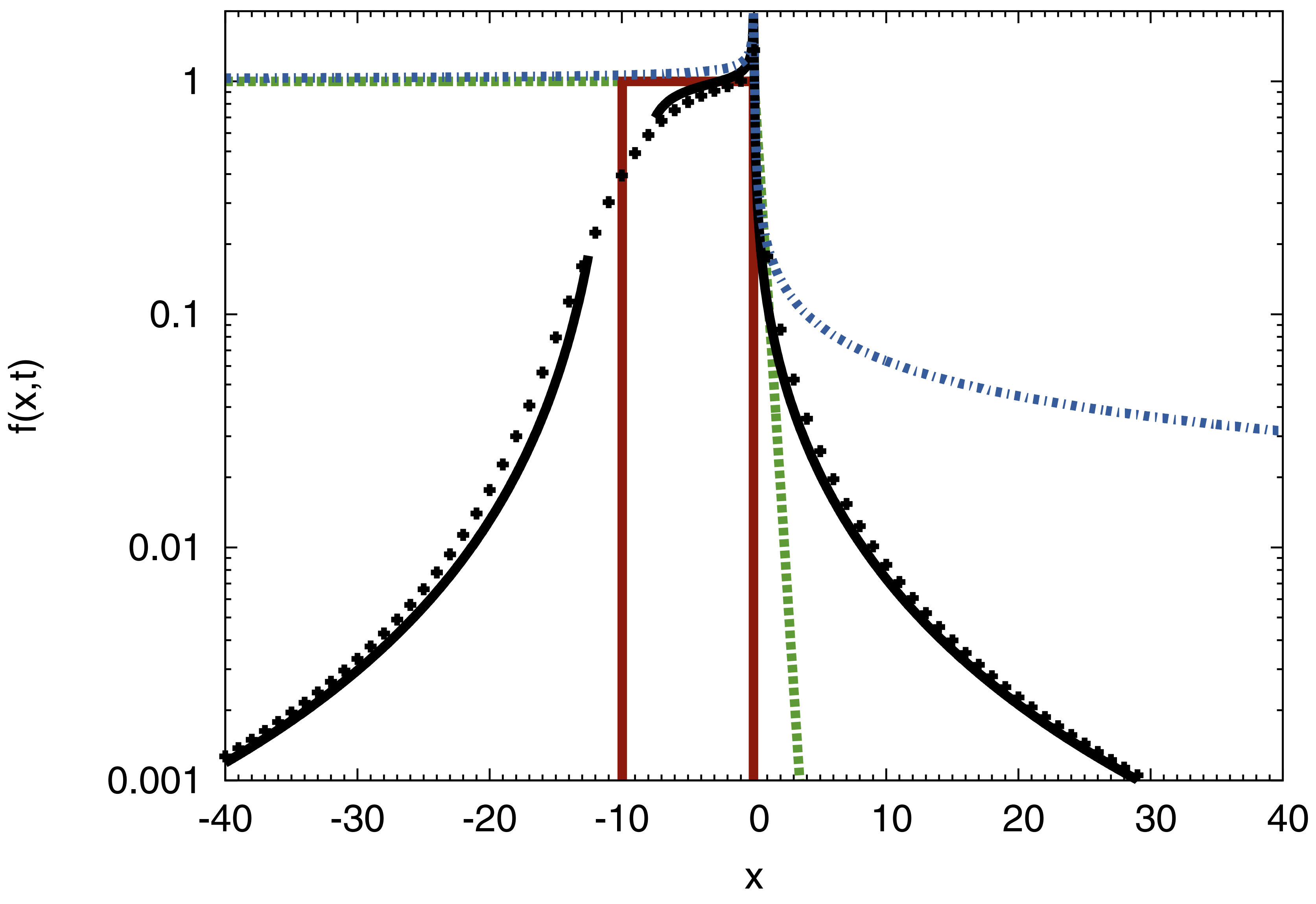

$x > 0$

$$f(x,t) \approx \frac{1}{\pi} \sin\left(\frac{\pi}{2}\alpha\right) \Gamma(\alpha-1) \frac{\kappa}{a^2} \left[x^{1-\alpha} - \frac{x+\alpha a t}{(x+at)^{\alpha}} \right]$$

$$f(x,t) \approx \frac{1}{\pi} \sin\left(\frac{\pi}{2}\alpha\right) \Gamma(\alpha-1) \frac{\kappa}{a^2} x^{1-\alpha}, \quad 0 < x \ll at.$$

$x < 0$

$$f(x,t) \approx \frac{1}{2\pi} \sin\left(\frac{\pi}{2}\alpha\right) \Gamma(1+\alpha) \frac{\kappa t^2}{|x|^{1+\alpha}}, \quad |x| \gg at,$$

Models for Anomalous Diffusion

PHYSICAL REVIEW E **75**, 056702 (2007)

Competition between subdiffusion and Lévy flights: A Monte Carlo approach

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(Received 1 March 2007; published 7 May 2007)

The general form of the celebrated fractional Fokker-Planck equation (FFPE), describing the competition between subdiffusion and Lévy flights under the influence of an external potential $V(x)$, is given in [1]:

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left(\frac{\partial}{\partial x}\frac{V'(x)}{\eta} + K\nabla^{\mu}\right)w(x,t). \qquad (1)$$

Here, the operator

$${}_0D_t^{1-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}f(s)ds, \qquad (2)$$

$0<\alpha<1$, stands for the fractional derivative of the Riemann-Liouville type and ∇^{μ}, $0<\mu\leq 2$, is the Riesz fractional derivative with the Fourier transform $\mathcal{F}\{\nabla^{\mu}f(x)\} = -|k|^{\mu}\tilde{f}(k)$ [5]. The occurrence of the operator ${}_0D_t^{1-\alpha}$ in Eq.

In this paper, we derive the stochastic representation of the solution $w(x,t)$ of the FFPE (1)—i.e., we show that $w(x,t)$ is equal to the probability distribution function (PDF) $p(x,t)$ of the subordinated process

$$Y(t) = X(S_t). \qquad (3)$$

Here the parent process $X(\tau)$ is defined as the solution of the stochastic differential equation (SDE)

$$dX(\tau) = -V'(X(\tau))\eta^{-1}d\tau + K^{1/\mu}dL_{\mu}(\tau) \qquad (4)$$

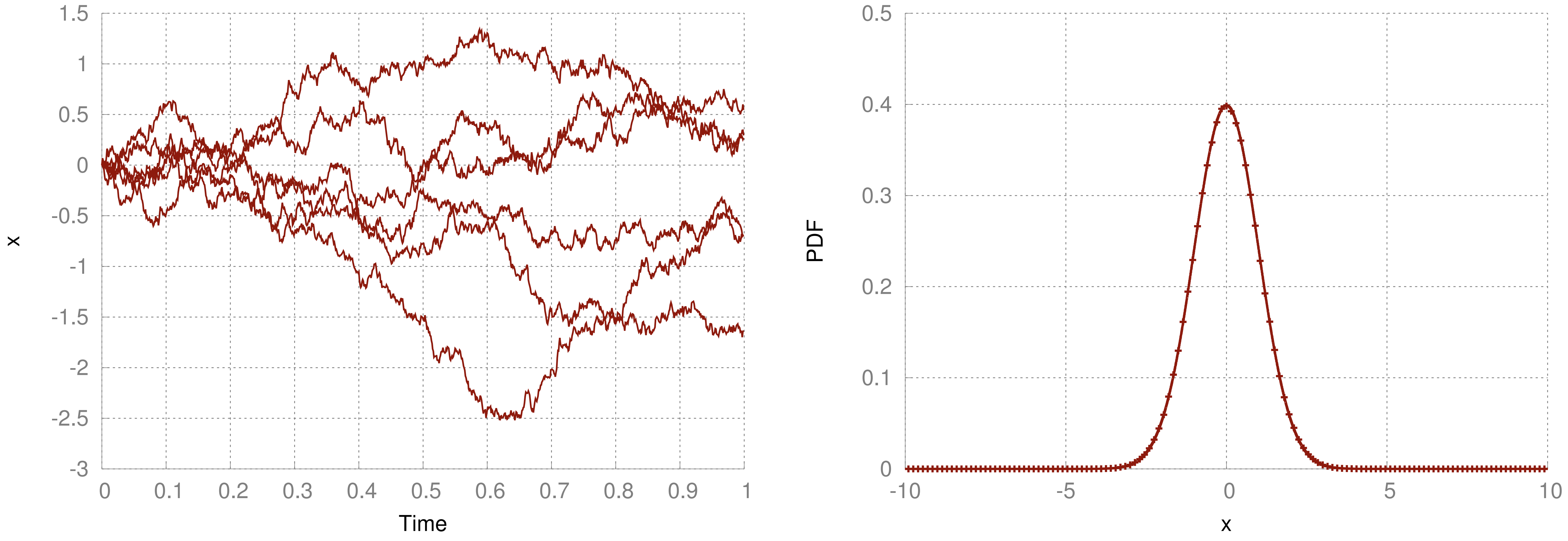
driven by symmetric μ-stable Lévy motion $L_{\mu}(\tau)$ with the Fourier transform $\langle e^{ikL_{\mu}(\tau)}\rangle = e^{-\tau|k|^{\mu}}$ [13]. Observe that $L_{\mu}(\tau)$ is

Models for Anomalous Diffusion

Lévy flights and subdiffusion

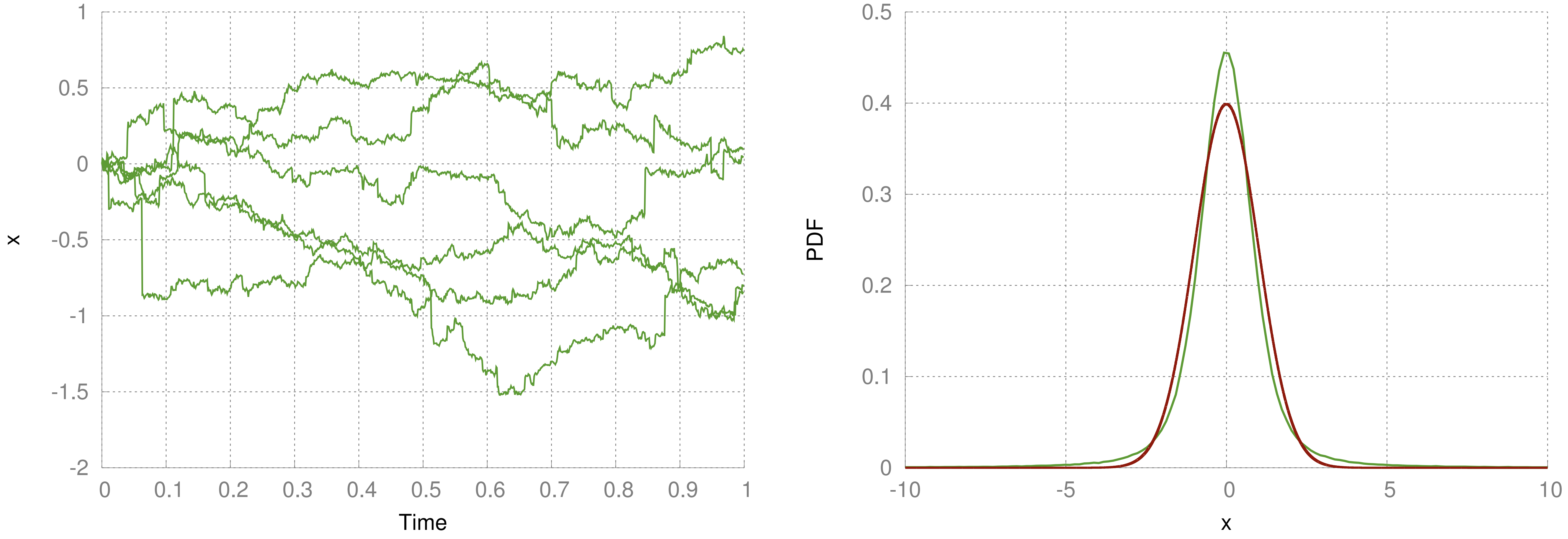
$\alpha = 1.0$

$\mu = 2.0$



$\alpha = 1.0$

$\mu = 1.5$

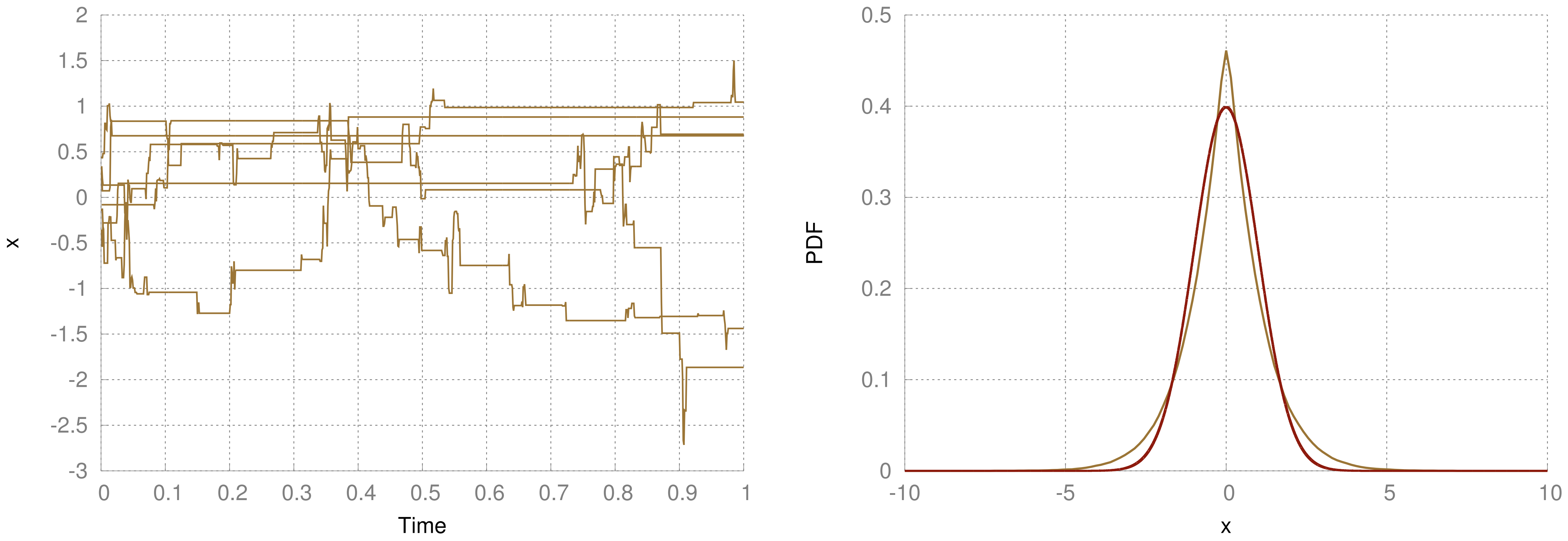


Models for Anomalous Diffusion

Lévy flights and subdiffusion

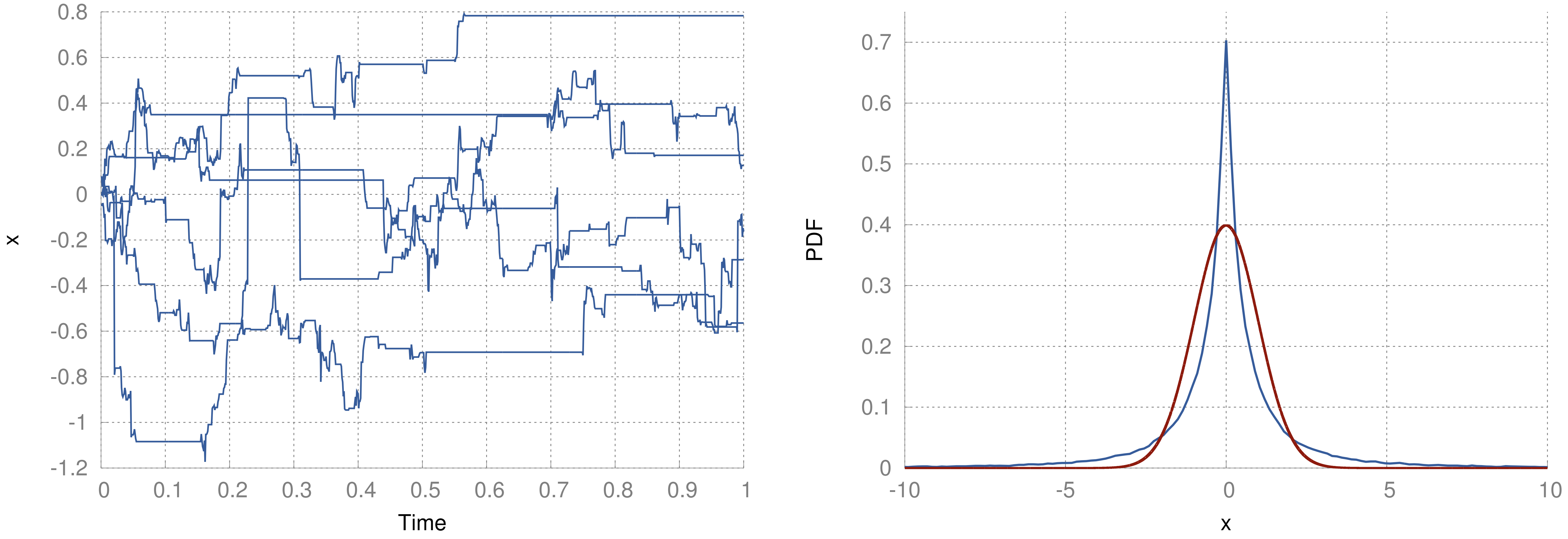
$\alpha = 0.7$

$\mu = 2.0$



$\alpha = 0.7$

$\mu = 1.2$



Observations at Shocks

Solar Wind termination shock (Perri & Zimbardo 2009)

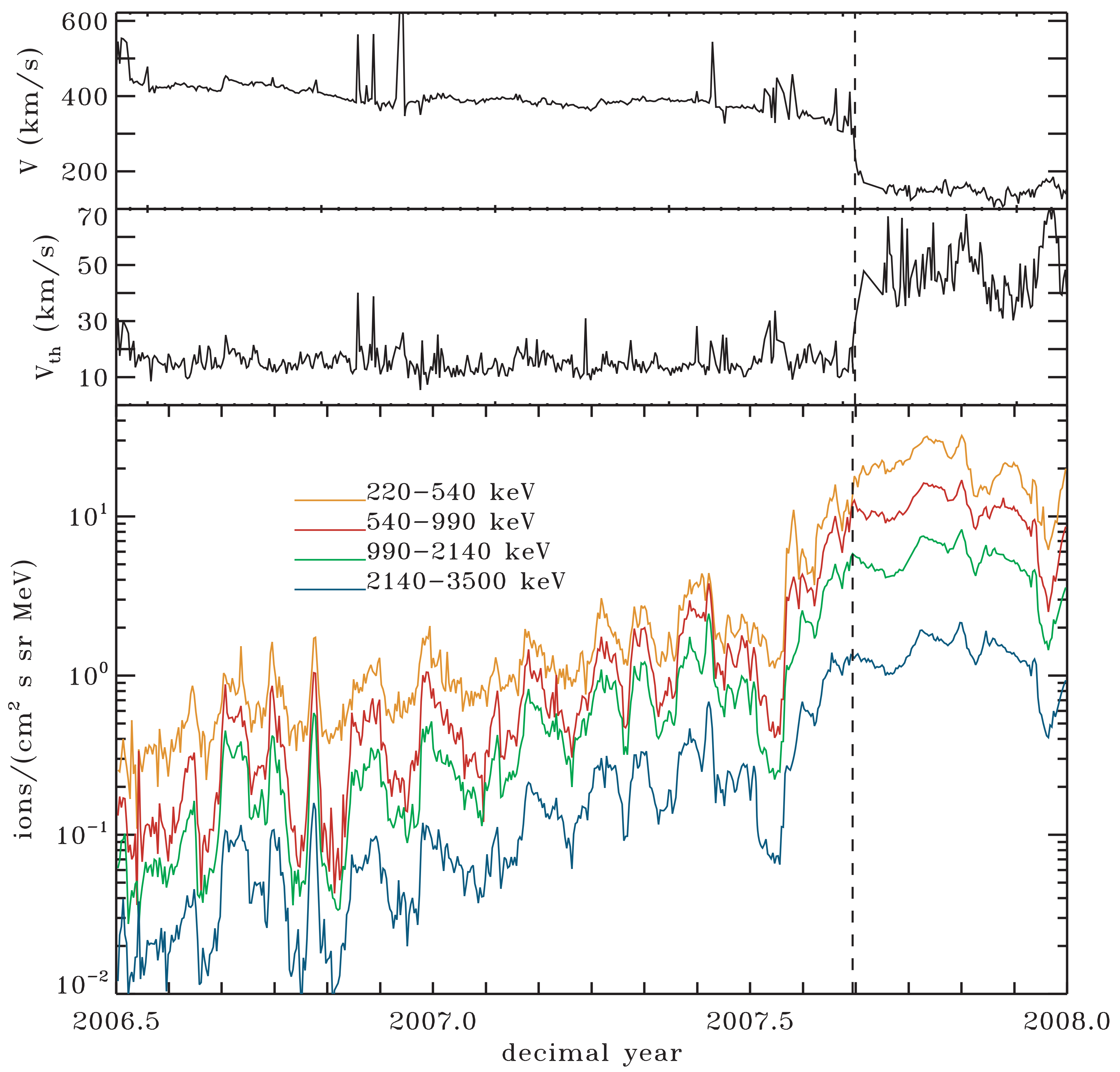


Figure 1. Upper panels show the time evolution of the magnitude of the proton bulk velocity and the proton thermal speed measured by the PLS instrument on board V2 (P.I.: J. Richardson); the lower panel displays the energetic particle data measured by the LECP instrument on board V2 (P.I.: S. M. Krimigis). The shock crossing time is indicated by a vertical dashed line.

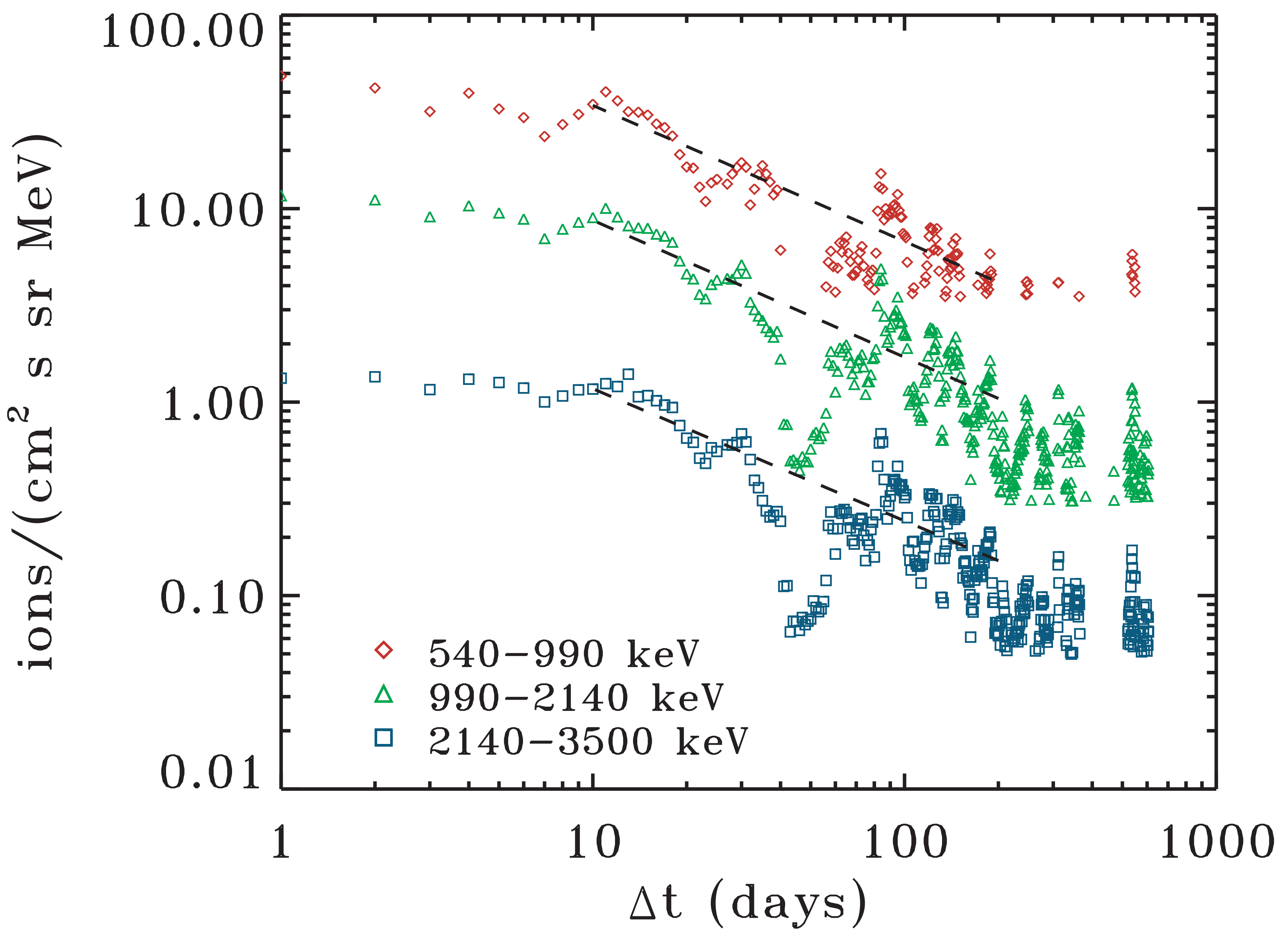


Figure 2. Power-law fits of the energetic particle fluxes in various energy channels.

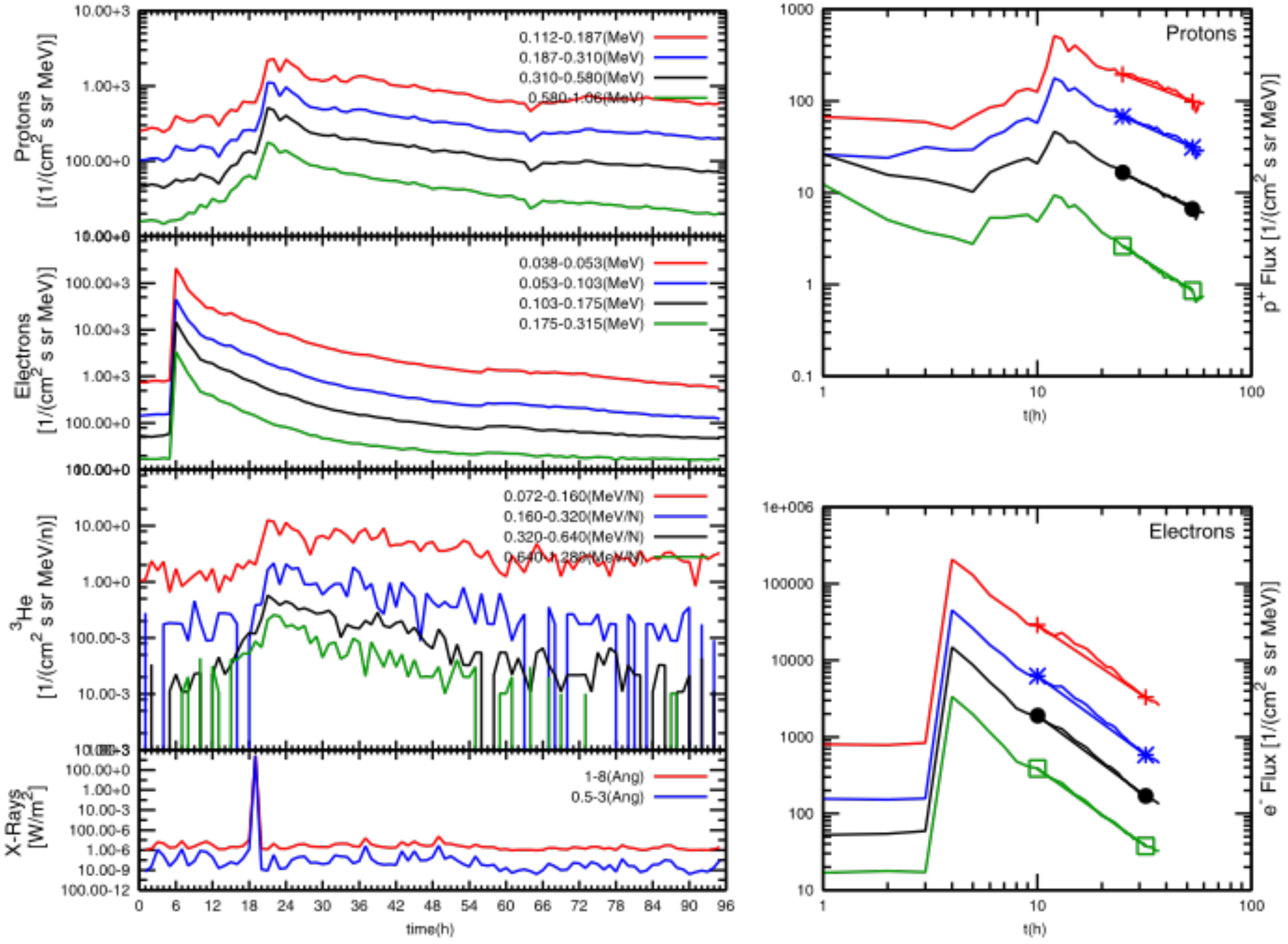
Table 1

Fit Parameters for the Ion Time Profiles at the Termination Shock

Energy (keV)	γ	α	χ^2_{pl}	χ^2_e
540–990	0.70 ± 0.07	1.30	0.22	0.40
990–2140	0.71 ± 0.08	1.29	0.18	0.25
2140–3500	0.68 ± 0.15	1.32	0.05	0.07

Observations at Shocks

Solar energetic particles (Trotta & Zimbardo 2011)



20 Feb 2002

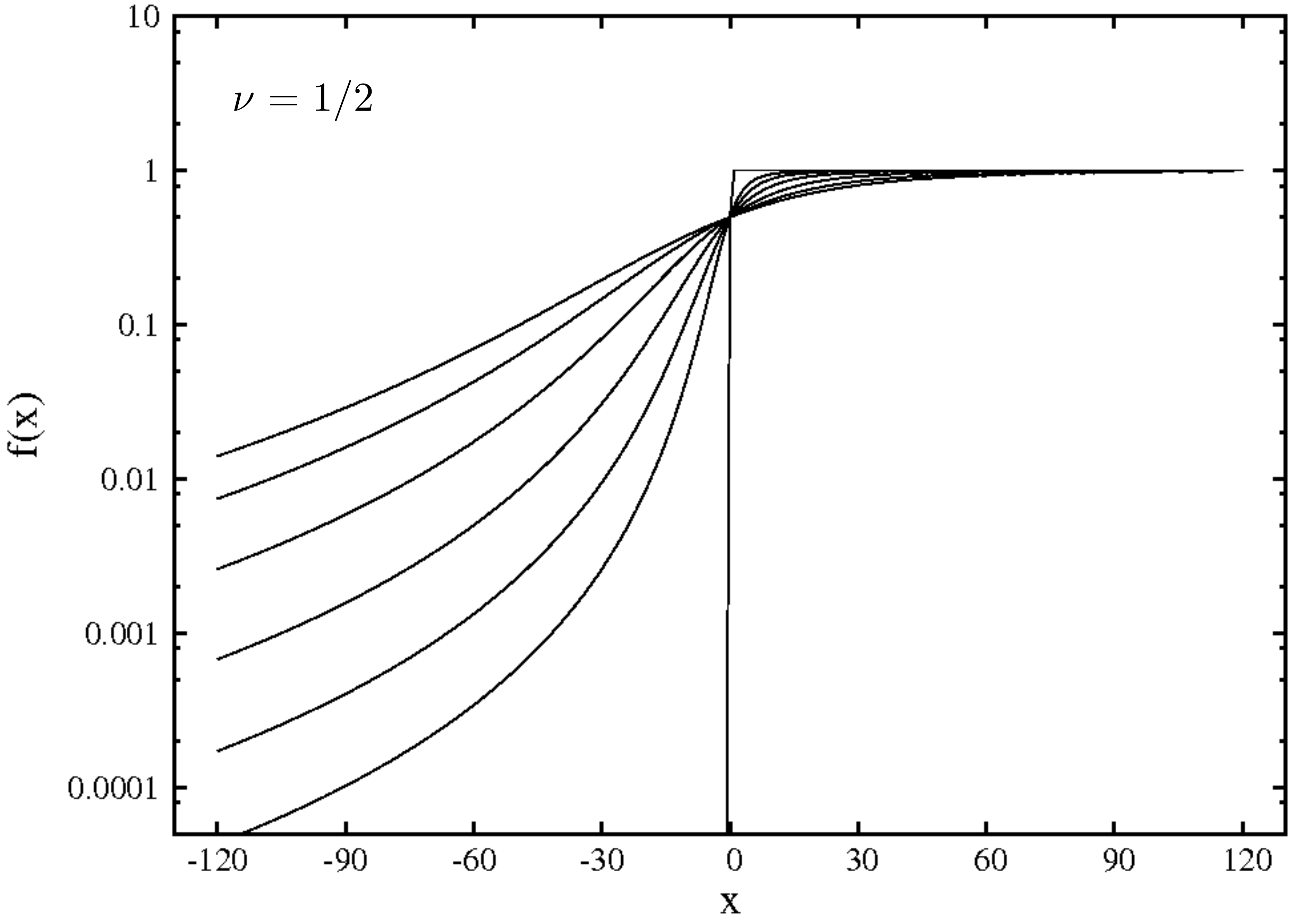
Non-linear diffusion at shocks

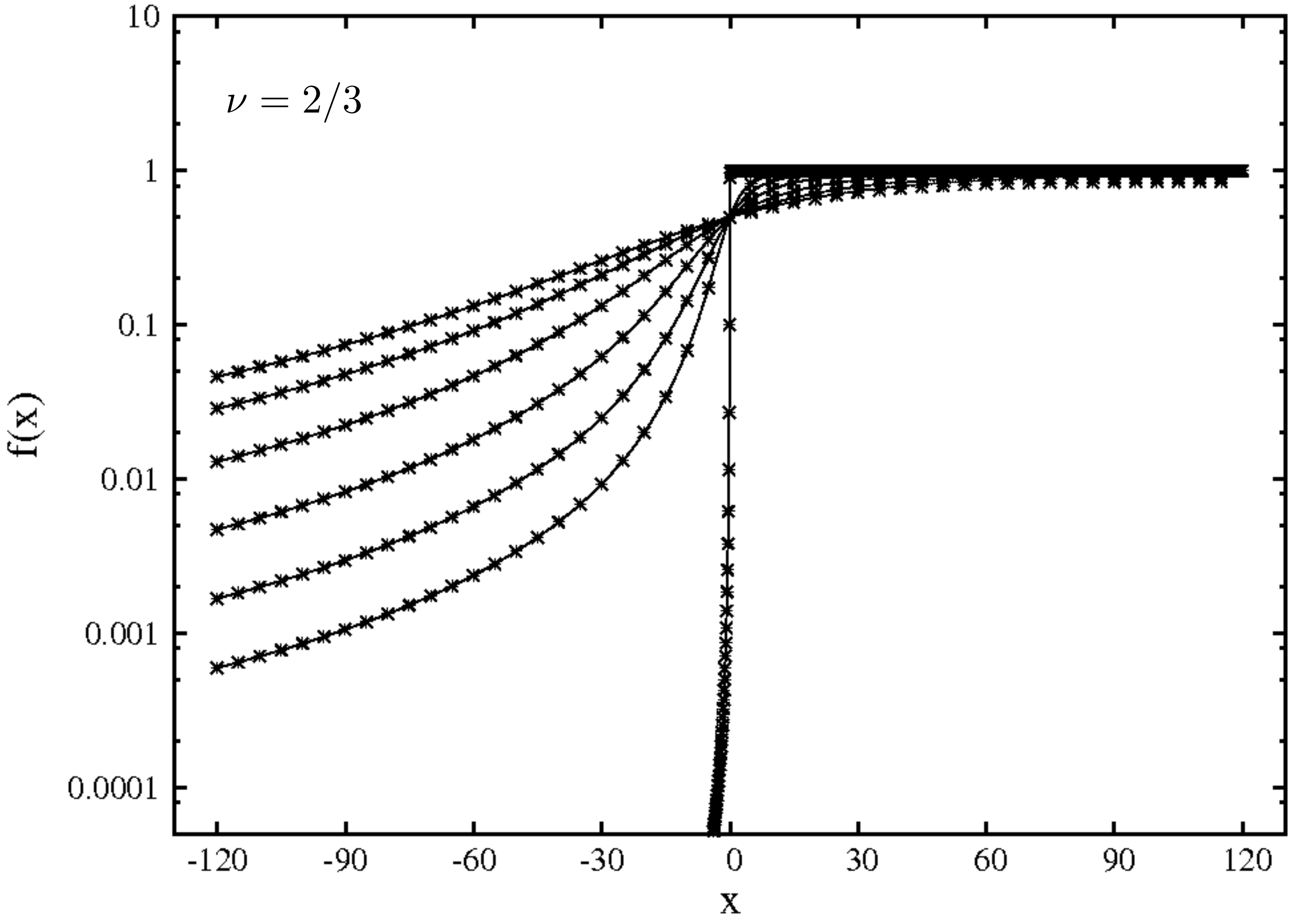
Litvinenko, Fichtner et al. 2016

$$D = D_0 \left| \frac{\partial f}{\partial x} \right|^{-\nu}, \quad D_0 = \mathrm{const.}$$

[Ptuskin et al., 2008]

$$\langle x^2 \rangle = 2t^{1/(1-\nu)} \int_0^\infty \xi^2 \phi(\xi) d\xi \sim t^{1/(1-\nu)}.$$





Summary

- Anomalous diffusion can occur in many natural and artificial systems
- Both super- and subdiffusion can show as transient or long-time persistent features.
- They can compete with each other simultaneously (need for higher moments)
- In the energetic particle context, there are potential applications to shocks, shock-acceleration, stochastic acceleration and particle transport
- The correct mathematical tools to describe anomalous diffusion need to be developed and tested, and supported by first-principle studies, e.g. particle tracing in turbulence

Thank you!